Statistics on Lie groups: can we obtain a consistent framework with pseudo-Riemannian metrics?

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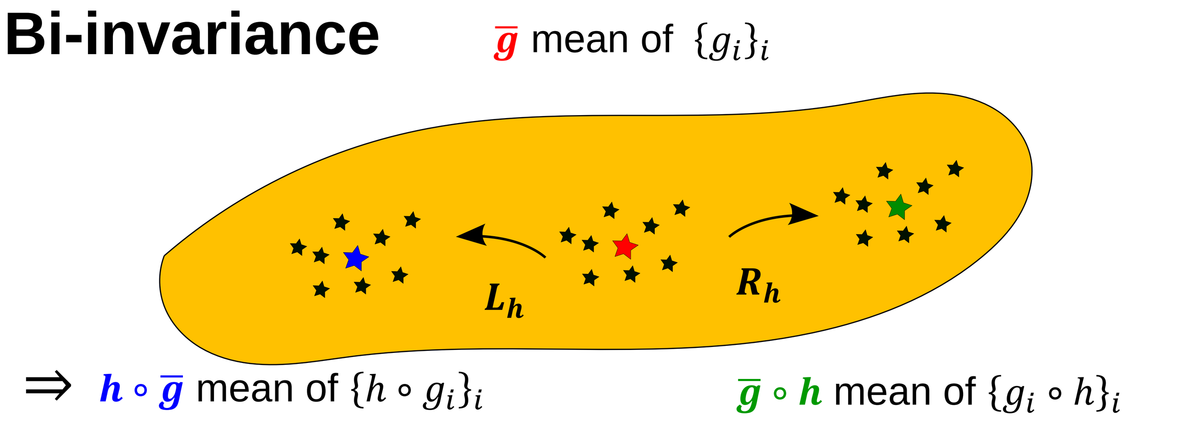
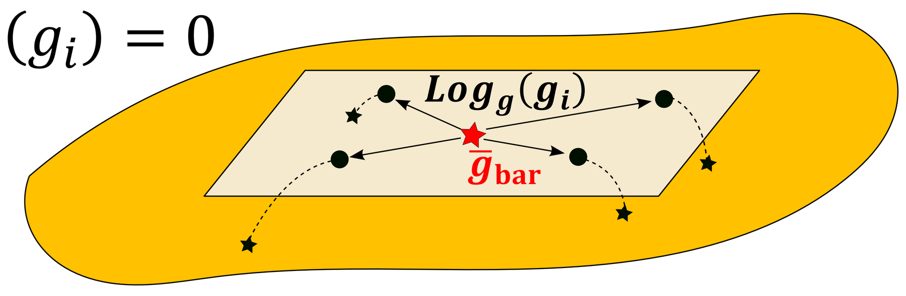
Models in computer vision often require to estimate *continuous transformations of the 3D space*, i.e. *elements of a Lie group*. This demands a *consistent* framework for statistics on Lie groups. It is known that there is no fully consistent (bi-invariant) Riemannian metric on most Lie groups [1]. We investigate here the existence of bi-invariant pseudo-Riemannian metrics and we present an algorithm to compute such a pseudo-metric on a Lie group if it exists. Unfortunately results show that most Lie groups do not possess a bi-invariant pseudo-metric in general, although the class of such groups is larger than for the Riemannian case.

Consistent statistical framework on Lie groups : Example of the mean

Requirement for a "good" definition of mean:
Bi-invariance



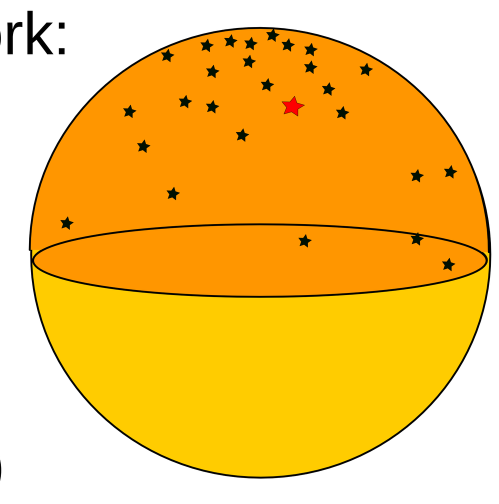
Group exponential barycenter:
Solution $\bar{g}_{\mathbf{bar}}$ of the barycentric equation:
$\sum_i Log_g(g_i) = 0$



where Log is the group logarithm.

Computational framework:

Characterize the domain where we Can do statistics (e.g. domain of existence and uniqueness of $\bar{g}_{\mathbf{bar}}$)

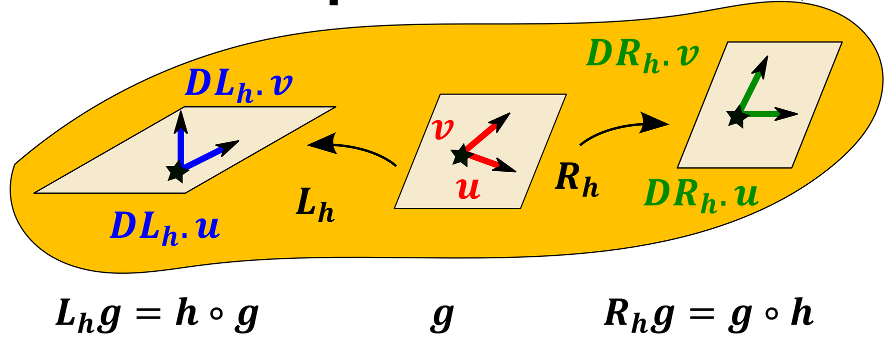


Question 1: which Lie groups do we add by asking for a bi-invariant pseudo-metric instead of a bi-invariant metric?
Question 2: is the pseudo-Riemannian setting rich enough to provide a consistent framework for statistics on Lie groups?

An algorithm to compute bi-invariant pseudo-metrics on Lie groups

Definitions

Lie algebra $\mathfrak{g} = \left(\mathrm{T_e G}, +, ., [,]_{\mathfrak{g}}\right)$
Bi-invariant pseudo-metric $<,>$



$< [x,y]_{\mathfrak{g}}, z >_e + < y, [x,z]_{\mathfrak{g}} >_e = 0$

Adjoint representation
$ad\colon \mathfrak{g} \mapsto \mathfrak{gl}(\mathfrak{g})\,,\ x \mapsto ad(x) = [x,\cdot]_{\mathfrak{g}}$

Adjoint representation decomposition
$\mathfrak{g} = \mathrm{B}_1^{\mathfrak{g}} \oplus_{\mathfrak{g}} \ldots \oplus_{\mathfrak{g}} \mathrm{B}_\mathrm{N}^{\mathfrak{g}}$, where:

- $\mathfrak{g} = \mathrm{B}_1^{\mathfrak{g}} \oplus \cdots \oplus \mathrm{B}_\mathrm{N}^{\mathfrak{g}}$ as vector spaces
- $\forall i,\ \left[\mathfrak{g}, \mathrm{B}_i^{\mathfrak{g}}\right] \subset \mathrm{B}_i^{\mathfrak{g}}$, i.e. $\mathrm{B}_i^{\mathfrak{g}}$an ideal

Double extension $B = W \oplus S \oplus S^*$

- $B = W \oplus S \oplus S^*$ as vector spaces
- S is a simple Lie subalgebra of B
- S^* is the co-adjoint representation of S
- W is a Lie algebra, and a S-representation
- $[w,w']_B = [w,w']_W + \beta(w,w')$

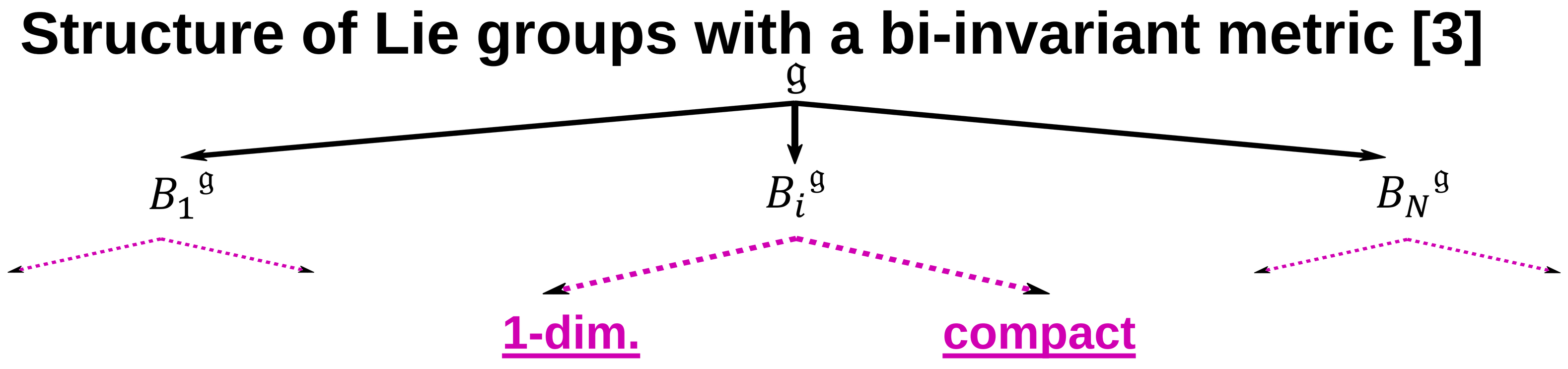
where $\beta\colon \Lambda^2 W \mapsto S^*$ is an equivariant map

Theorem

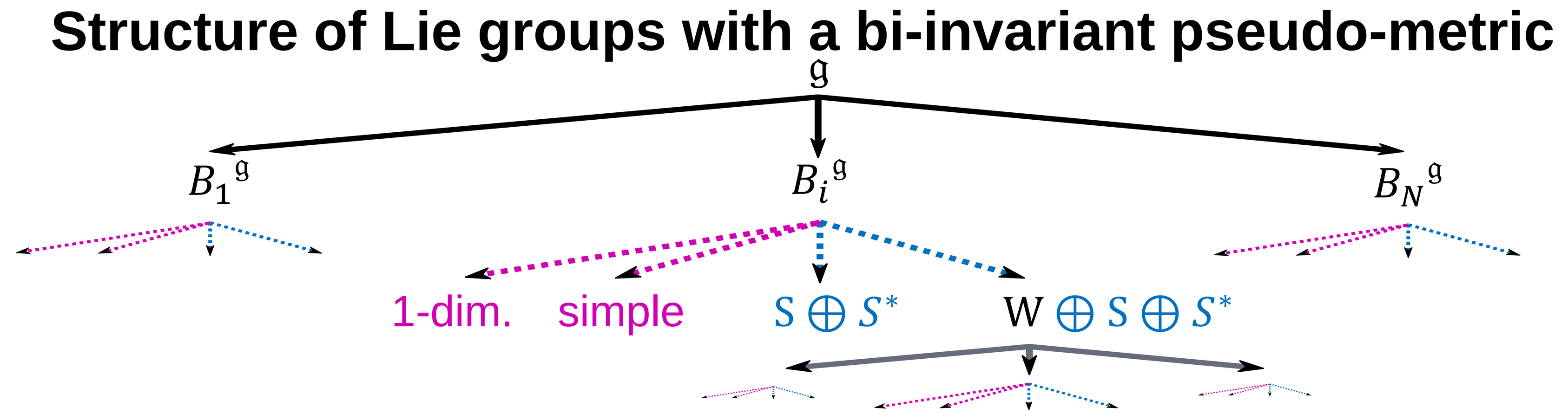
Theorem [2]: Structure of a Lie algebra that admits a bi-invariant pseudo-metric
$\mathfrak{g}$ has a bi-invariant pseudo-metric iff its adjoint representation decomposition $\mathfrak{g} = \mathrm{B}_1^{\mathfrak{g}} \oplus_{\mathfrak{g}} \ldots \oplus_{\mathfrak{g}} \mathrm{B}_\mathrm{N}^{\mathfrak{g}}$ has indecomposable $\mathrm{B}_i^{\mathfrak{g}}$ of:
Type (1): $B_i^{\mathfrak{g}}$ simple or 1-dim. or of Type (2): $B_i^{\mathfrak{g}} = \mathrm{W} \oplus \mathrm{S} \oplus \mathrm{S}^*$ double extension.

Comparison

Structure of Lie groups with a bi-invariant metric [3]



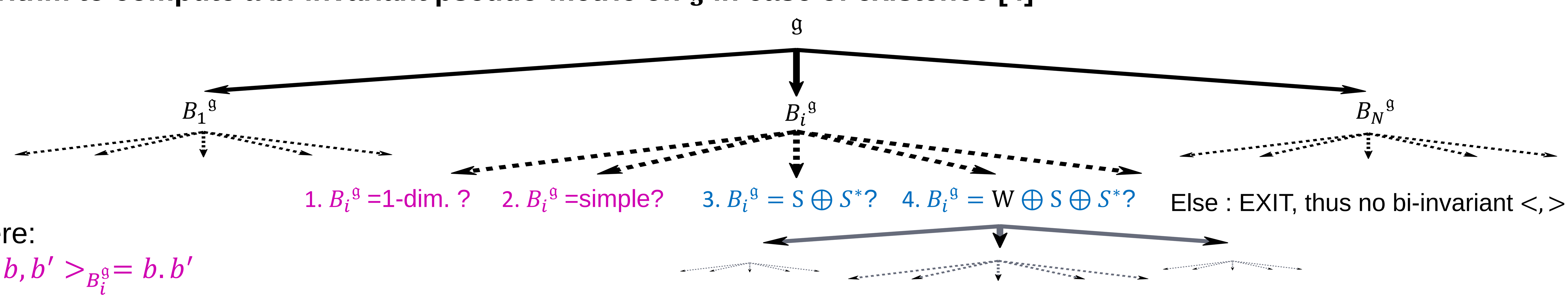
Structure of Lie groups with a bi-invariant pseudo-metric



Answer 1: By asking for a pseudo-metric instead of a metric, we add the simple non compact case and the double extension case.

Algorithm

Algorithm to compute a bi-invariant pseudo-metric on $\mathfrak{g}$ in case of existence [4]

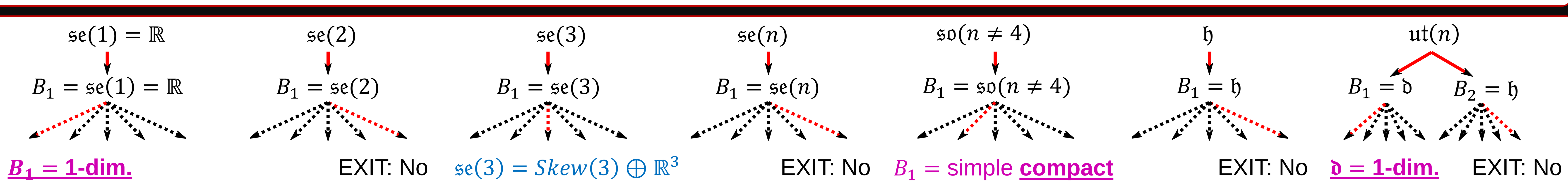


Where:

1. $< b, b' >_{B_i^{\mathfrak{g}}} = b.b'$
2. $< b, b' >_{B_i^{\mathfrak{g}}} = Killing(b, b')$
3. $< s+f, s'+f' >_{B_i^{\mathfrak{g}}} = f(s') + f'(s)$
4. $< w+s+f, w'+s'+f' >_{B_i^{\mathfrak{g}}} = < w, w' >_W + f(s') + f'(s)$

And recombination on $\mathfrak{g}$:
$< b_1 + \cdots + b_N, b'_1 + \cdots + b_N >_{\mathfrak{g}} = < b_1, b'_1 >_{B_1{}^{\mathfrak{g}}} + \cdots + < b_N, b'_N >_{B_N{}^{\mathfrak{g}}}$

Results



Answer 2: The class of Lie groups with bi-invariant pseudo-metric is not large enough to try using the pseudo-Riemannian setting [4].

References: [1] Pennec & Arsigny, *Exponential Barycenters of the canonical Cartan-Schouten connection and Invariant means on Lie groups* (2012), [2] Medina & Revoy, *Caractérisation des groupes de Lie ayant une pseudo-métrique bi-invariante,* (1982), [3] Cartan, *La théorie des groups finis et continus et l'analyse situs,* (1952), [4] Miolane & Pennec, *Statistics on Lie groups: a need to go beyond the pseudo-Riemannian framework,* (2015).